Analytic-Calculus Duration and Convexity

For analytic financial valuation functions, the value impact of a small rate change, \(d\), can be evaluated by Taylor Series:

\[
B(R + d) = B(R) + B'(R)d + \frac{B''(R)}{2}d^2 + \frac{B^{(3)}(R)}{6}d^3 + \ldots
\]

The expansion coefficients for each term are \(\frac{B^{(n)}(R)}{n!}d^n\).

For finance applications, the first two terms are most important: Duration and Convexity.\(^1\)

\[
\text{Duration} = \Delta = -\frac{B'(R)}{B(R)}
\]

\[
\text{Convexity} = \Gamma = \frac{B''(R)}{B(R)}
\]

\(^1\) We treat a set of durations that include Modified Duration. We don’t treat maturity-weighted or MacCauley duration. MacCauley duration is defined as follows:

\[
\sum_{i=1}^{n} \frac{PV_i}{V},
\]

where \(PV_i\) is the present value of the period \(i\) cashflow, and \(V\) is the bond value. With MacCauley duration, cashflow present values are weighted by the associated maturity and summed.
The review covers two types of valuation: market rates and corresponding continuously compounded rates. Two security types are considered: zero-coupon and a two-period coupon paying note.

**Market-Rate Discounting**

For maturities less than a year, the Discount Bond Formula that provides value is the following: 

\[
\frac{1}{1+R \cdot t}.
\]

The first example will have the interest rate, \( R \), at \( 5\% = 0.05 \), and maturity of one year, \( t=1 \). At these parameter values, the value of the bond, \( B^0 \), is roughly 0.95 (a 5% discount), or, more exactly, \( B^0 = \frac{1}{1+0.05 \cdot 1} = 0.952381 \).

For longer maturities, the semi-annual bond equivalent basis is most frequently used:

\[
\frac{1}{(1+R/2)^{2t}}.
\]

Though a bit more complicated, conducting the analysis at hand for longer-term securities is practically the same as the short-term case.
The derivative or change of the discount bond value can be calculated in many ways. The simplest and clearest is to simply change the variable of interest, the interest rate, and see how the discount bond value changes. It’s best to change rates only a little, and our change will be one basis point (bp), which is d=0.01%=0.0001.

At a slightly higher interest rate level, \( R + d = 5\% + 0.01\% = 5.01\% = 0.05 + 0.0001 = 0.0501 \), the discount bond value falls: \( B^+ = \frac{1}{1 + 0.0501 \times 1} = 0.95229 \).

The derivative or rate of change of the bond value is approximated with the bond value at the higher rate minus the bond value at the original rate. This value-change numerator is divided by the rate change:

\[
d^+ = \frac{B^+ - B^0}{d} = \frac{0.95229 - 0.952381}{0.0001} = \frac{-0.00009069431}{0.0001} = -0.906943
\]

It is not our objective to calculate approximate bond values. Nevertheless, we can check how a linear approximation to the bond value performs given a small
change in interest rates. This linear approximation uses only the first term in the value Taylor series expansion.

The $d^+$ linear approximation to the bond price decrease is $B^0 + (d^+ \cdot d) = 0.952381 - 0.906943 \cdot 0.0001 = 0.952381 - 0.0000906943 = 0.95229$. This linearly approximated value approximately equals the actual value at a 5.01% rate, 0.95229.

As discussed in the main section of the interest rate review notes, a key quantity that provides information on relative bond value sensitivity to interest rate changes is duration. Duration is simply the negative of the bond interest rate derivative divided by the bond price. This weighting adjustment of duration allows comparison of the interest rate sensitivity of bonds with different principal amounts.

$$\Delta^+ = -\frac{d^+}{B^0} = -\frac{-0.906943}{0.952381} = 0.95229$$

An intuitive interpretation of duration is the relative interest rate sensitivity to a one-year maturity continuously compounded zero-coupon bond. (As we will see
below, analytic/calculus- based continuously compounded zero-coupon bond durations equal the associated bond maturity.)

Therefore in our example, a one-year zero-coupon bond discounted on a simple interest basis has a duration that is roughly 95% of the one-year maturity continuously compounded zero-coupon bond (which also has one-year duration).

We may also approximate the bond duration by the value change induced by a small rate decrease: \( R - d = 5\% - 0.01\% = 4.99\% = 0.05 - 0.0001 = 0.0499 \), and

\[
B^- = \frac{1}{1 + 0.0499 \times 1} = 0.952472.
\]

\[
d^- = \frac{B^0 - B^-}{d} = \frac{0.952381 - 0.952472}{0.0001} = \frac{-0.0000907116}{0.0001} = -0.907116.
\]

For this set of values, the \( d^- \) linear approximation to the bond price decrease is

\[
B^0 + d^- \times d = 0.952381 - 0.907116 \times 0.0001 = 0.952381 - 0.0000907116 = 0.952472.
\]

This linearly approximated value is very close to the actual value at a 4.99\% rate, 0.952472.
Duration with a small rate decrease follows:

\[
\Delta^- = -\frac{d^-}{B^0} = -\frac{-0.907116}{0.952381} = 0.952472
\]

Convexity is a measure of the acceleration or deceleration of bond value changes to changes in interest rates:

\[
\text{Convexity} = \frac{\Delta^- - \Delta^+}{\Delta} = \frac{0.952472 - 0.95229}{0.0001} = 1.81406
\]

In the present case, convexity is positive because the negative slope of the bond value as a function of rates falls in absolute value as rates increase. Between 4.99% (rate down) and 5%, the approximate slope, \(d^-\), is -0.907116. Between 5.0% (rate down) and 5.01%, the approximate slope, \(d^+\), is -0.906943. Therefore, the slope increases by becoming less negative at higher interest rates.

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2 This convexity definition is the negative of the bond value second derivative, which is the more normal convexity definition in fixed income or 1.78182 in this case. We use this alternative definition to simply link duration and convexity.
The analytic or calculus-based measures of duration and convexity follow from the derivative chain rule: 
\[
\frac{d(B(R)^n)}{dR} = nB(R)^{n-1}\frac{dB(R)}{dR},
\]
where \(B(R)\) is the bond value formula as a function of interest rates:
\[
B(R) = \frac{1}{1 + R \times t} = (1 + R \times t)^{-1}.
\]

For the first derivative,
\[
\frac{dB(R)}{dR} = -1 \times (1 + R \times t)^{-2} \times t = \frac{-t}{(1 + R \times t)^2},
\]
and with \(R=0.05\) and \(t=1\),
\[
\frac{dB(R)}{dR} = \frac{-1}{(1 + 0.05 \times 1)^2} = -0.907029.
\]

This analytic first derivative falls between the two discrete measures, which were calculated previously: \(d^+ = -0.906943\) and \(d^- = -0.907116\). Furthermore, the continuous value is equal to the average of the two discrete measures out to nine digits.
For analytic convexity, we first calculate the second derivative:

\[
\frac{dB^2(R)}{dR^2} = \frac{d}{dR} \left( \frac{dB(R)}{dR} \right) = \frac{d}{dR} \left( \frac{-t}{(1+R*t)^2} \right) = \frac{d}{dR} \left( -t \cdot (1+R*t)^{-2} \right) = 2t^2 \cdot (1+R*t)^{-3}.
\]

For the benchmark case (R=5% and t=1),

\[
\frac{dB^2(R)}{dR^2} = \left( 2 \cdot (1+0.05*1)^{-3} \right) = \frac{2}{(1+0.05*1)^3} = 1.72768
\]

Convexity, analogous to duration, is the value-weighted second derivative:

\[
\text{Convexity} = \frac{dB^2(R)}{dR^2} \cdot B^0 = \frac{2t^2}{(1+R*t)^3} \cdot \frac{1}{(1+R*t)^2} = \frac{2t^2}{(1+R*t)^2}
\]

With R = 0.05 and t = 1, \( \text{Convexity} = \frac{2}{(1+0.05*1)^2} = 1.81406 \)

For our current case, analytic convexity equals discrete convexity out to eight digits.
Coupon-Paying Bonds

The rate review notes cover discrete approximations to a coupon-paying bond duration and convexity. The analytic case is conceptually equivalent to the zero coupon case. However, a separate derivative needs to be calculated for each bond cash flow, and these derivatives are summed to get the total first and second derivatives for coupon-paying securities.

The simplest case is the one-year maturity note paying semiannual coupon interest.

\[ B_1 = \frac{C_1/2}{(1 + R_{1/2}/2)} + \frac{C_1/2 + 1}{(1 + R_1)} \]

To treat the simplest case of parallel shifts in the term structure, we introduce a small rate change variable, \( q \), which is (initially) zero:

\[ B_1 = \frac{C_1/2}{(1 + (R_{1/2} + q)/2)} + \frac{C_1/2 + 1}{(1 + R_1 + q)} = C_1/2 \left(1 + (R_{1/2} + q)/2\right)^{-1} + \left(C_1/2 + 1\right) \left(1 + R_1 + q\right)^{-1} \]
To differentiate, we again use the chain rule:

\[
\frac{dB_1}{dq} = -\frac{C_1}{4(1+(q+R_{1/2})/2)^2} - \frac{(1+C_1/2)}{(1+q+R_1)^2} = \frac{-C_1}{4(1+(q+R_{1/2})/2)^2} - \frac{(1+C_1/2)}{(1+q+R_1)^2}
\]

For the example parameter values, \(C_1=4.93902\%\), \(R_{1/2}=4.93902\%\), and \(R_1=5.0\%\), the derivative value is -0.941188. Since the bond is a $1 principal par-value bond, the bond value is a dollar. Therefore, duration equals minus one times the derivative, 0.941188. As in the case of the zero coupon bond example, the analytic/calculus duration value falls between the discrete rate increase duration, 0.9411, and discrete rate decrease duration, 0.9413, cases.

### Continuously-Compounded Rate Discounting

For all maturities, the Discount Bond Formula is the following: \(e^{-Zt}\). The first example sets the interest rate, \(Z\), at 4.879\%=0.04879 for all maturities. The
continuously compounded rate is consistent with 5% one-year and 4.939% half-year simple market interest rates.

Initially, we set maturity at one year, $t=1$. At these parameter values, the value of the bond, $B^0$, is, as in the market-rate case, $B^0 = e^{-0.04879} = 0.952381$.

The derivative or change of the discount bond value is approximated for a small rate change, one basis point (bp): which is $d=0.01\% = 0.0001$.

At a slightly higher interest rate level, $Z + d = 4.879\% + 0.01\% = 4.889\% = 0.04879 + 0.0001 = 0.04889$, the discount bond value falls: $B^+ = e^{-0.04889*1} = 0.95229$.

The derivative or rate of change of the bond value is approximated with the bond value at the higher rate minus the bond value at the original rate. This value-change numerator is divided by the rate change:

$$d^+ = \frac{B^+ - B^0}{d} = \frac{0.95229 - 0.952381}{0.0001} = \frac{-0.0000952333}{0.0001} = -0.952333$$
Duration is the $d^+$ quantity divided by the bond price. This weighting adjustment of duration allows comparison of the interest rate sensitivity of bonds with different principal amounts.

$$\Delta^+ = -\frac{d^+}{B^0} = -\frac{-0.952333}{0.952381} = 0.99995$$

We may also approximate the bond duration by the value change induced by a small rate decrease: $Z - d = 4.879\% - 0.01\% = 4.869\% \approx 0.04879 - 0.0001 = 0.04869$, and $B^- = e^{-0.04869} = 0.952476$.

$$d^- = \frac{B^0 - B^-}{d} = \frac{0.952381 - 0.952476}{0.0001} = -\frac{-0.0000952429}{0.0001} = -0.952429$$

Duration with a small rate decrease follows:

$$\Delta^- = -\frac{d^-}{B^0} = -\frac{-0.952429}{0.952381} = 1.00005$$

As in the market-rate case, the average of the two approximate durations is very close to the correct value, 1.0.
With regard to convexity:

\[
\text{Convexity} = \frac{\Delta^- - \Delta^+}{\Delta} = \frac{1.00005 - 0.99995}{0.0001} = 1.0
\]

The analytic or calculus-based measures of duration and convexity follow from the exponential differentiation rule:

\[
\frac{dB(Z)}{dZ} = \frac{d(B(Z))}{dZ} e^{B(Z)}, \text{ where } B(Z) \text{ is the bond value formula as a function of interest rates: } B(Z) = e^{-zt}.
\]

For the first derivative, \( \frac{dB(R)}{dR} = -te^{-Zt} = tB(Z) \), and with \( Z=0.04879 \) and \( t=1 \),

\[
\frac{dB(R)}{dR} = -1e^{-0.04879*1} = -0.952381. \text{ Dividing by the bond value, } B(Z), \text{ duration equals maturity, or one.}
\]

For analytic convexity, we first calculate the second derivative:

\[
\frac{dB^2(Z)}{dZ^2} = \frac{d\left(\frac{dB(Z)}{dZ}\right)}{dZ} = d\left(\frac{te^{-Zt}}{dR}\right) = t^2 e^{-Zt} = t^2 B(Z).
\]
For the benchmark case ($Z=0.4879\%$ and $t=1$),

$$\frac{dB^2(Z)}{dZ^2} = t^2 e^{-zt} = 1e^{-0.048791} = 0.952381$$

**Coupon-Paying Bonds**

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The simplest case is the one-year maturity note paying semiannual coupon interest.

$$B_1 = C_1/2 e^{-Z_{1/2}^2/2} + (1+C_1/2)e^{-Z_1}$$

To treat the simplest case of parallel shifts in the term structure, we introduce a small rate change variable, $q$, which is (initially) zero:
\[ B_1 = \frac{C_1}{2} e^{-\left(\frac{Z_1}{2} + q\right)/2} + \left(1 + \frac{C_1}{2}\right) e^{-\left(Z_1 + q\right)} \]

Differentiating,

\[ \frac{dB_1}{dq} = \frac{d}{dq}\left(\frac{C_1}{2} e^{-\left(\frac{Z_1}{2} + q\right)/2} + \left(1 + \frac{C_1}{2}\right) e^{-\left(Z_1 + q\right)}\right) = -\frac{C_1}{4} e^{-\left(\frac{Z_1}{2} + q\right)/2} - \left(1 + \frac{C_1}{2}\right) e^{-\left(Z_1 + q\right)} \]

For the example parameter values, \( C_1 = 4.93902\% \), \( Z_{1/2} = 4.879\% \), and \( Z_1 = 4.879\% \), the derivative value is -0.98795. Since the bond is a $1 principal par-value bond, the bond value is a dollar. Therefore, duration equals minus one times the derivative, 0.98795.

For convexity,

\[ \frac{d^2B_1}{dq^2} = \frac{d}{dq}\left(\frac{-C_1}{4} e^{-\left(\frac{Z_1}{2} + q\right)/2} - \left(1 + \frac{C_1}{2}\right) e^{-\left(Z_1 + q\right)}\right) = \frac{C_1}{8} e^{-\left(\frac{Z_1}{2} + q\right)/2} + \left(1 + \frac{C_1}{2}\right) e^{-\left(Z_1 + q\right)} \]

And, numerically,

\[ \frac{d^2B_1}{dq^2} = \frac{C_1}{8} e^{-\left(\frac{Z_1}{2} + q\right)/2} + \left(1 + \frac{C_1}{2}\right) e^{-\left(Z_1 + q\right)} = 0.04939 / 8 e^{-0.04879/2} + (1 + 0.04939/2) e^{-0.04879} = 0.981925 \]